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1985 J. Phys. A: Math. Gen. 18 L801

(http://iopscience.iop.org/0305-4470/18/13/012)

J. Phys. A: Math. Gen. 18 (1985) L801-L804. Printed in Great Britain

LETTER TO THE EDITOR

Remark on the lattice field description of the disordered two-dimensional Ising model

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Received 6 June 1985

Abstract. A lattice field model which is believed to describe the disordered Ising model in two dimensions is considered. We find a smooth behaviour of the quenched thermodynamic functions dependent on temperature.

The behaviour of many spin models in statistical mechanics at the phase transition point and nearby can presumably be described by certain field theories. For instance, the Ising model on a hypercubic lattice governed by the Hamiltonian

$$H=\frac{1}{T}\sum_{r,r'\in\Lambda} J_{rr'}S_rS_{r'} \qquad (\Lambda\subset Z^d) \tag{1}$$

at temperature T with

$$J_{rr'}=\begin{cases} J & \text{for } |r-r'|=1 \\ 0 & \text{for } |r-r'|\neq 1 \end{cases} \qquad S_r\in\{-1,1\}$$

is equivalent in this sense to the so-called Φ^4 model (cf [1]) at least for space dimensions $d\geqslant 3$. The Ising model is exactly solvable for $d=2$ (cf [2]). The corresponding field theory is therefore simply the model of free fermions on the Ising lattice (cf [3]). It is approximately given by the Hamiltonian

$$\hat{H}_0=\begin{pmatrix} m & \Delta_1+\mathrm{i}\Delta_2 \\ \Delta_1-\mathrm{i}\Delta_2 & m \end{pmatrix} \tag{2}$$

on the square lattice $\Lambda\subset Z^2$ with a temperature dependent mass

$$m=(T-T_c)/T \qquad T_c=\tfrac{1}{2}\log(\sqrt{2}+1) \tag{3}$$

and the lattice differential operator in the j direction Δ_j. Thermodynamic functions of the two-dimensional Ising model can now be expressed by means of the Hamiltonian $\hat{H}_0$. The singular part of the internal energy $E(T)$, for instance, is then

$$E(T)=\lim_{\Lambda\uparrow Z^2}\frac{1}{|\Lambda|}\sum_{r\in\Lambda}\sum_{\mu=1}^{2}(\hat{H}_0^{-1})_{r\mu,r\mu}=:\mathrm{Tr}(\hat{H}_0^{-1}) \tag{4}$$

and we find immediately from (2) the well known logarithmic singularity of the specific heat as

$$C(T)=\frac{\partial}{\partial T}E(T)\underset{T\sim T_c}{\sim}-\tfrac{1}{2}\log m^2=-\log(|T-T_c|/T_c). \tag{5}$$

It was suggested by several authors [4-6] that (bond or site) disorder in the two-dimensional Ising model corresponds to local random potentials in the fermion field model. The Hamiltonian (2) is then extended to

$$\hat{H} = \hat{H}_0 + V \tag{6}$$

with

$$V_{r\mu,r'\mu'} = V_r \delta_{rr'} \delta_{\mu\mu'} \qquad (r, r' \in \Lambda; \mu, \mu' \in \{1, 2\}). \tag{7}$$

$\{V_r\}$ are chosen as statistically independent and uniformly distributed random variables. The quenched thermodynamic function can be evaluated from the averaged internal energy.

We choose for our investigation periodic boundary conditions and the following distribution function:

$$P(V) = \frac{\gamma}{\pi} \sum_{j=1}^{n} p_j [\gamma^2 + (V - a_j)^2]^{-1} \tag{8}$$

with $p_j > 0$, $\Sigma_j p_j = 1$, $\gamma > 0$.

This is just a Lorentz distribution centred at a_1 when we set $n = 1$. A continuous version for a binary distribution is given by $n = 2$.

Now we define the averaged internal energy as

$$E(T) = \mathrm{Tr}\left(\lim_{\varepsilon \to 0} \langle (\hat{H} + \mathrm{i}\varepsilon D)^{-1} \rangle \right) \tag{9}$$

with

$$D_{r\mu,r'\mu'} = D_r \delta_{rr'} \delta_{\mu\mu'} \qquad D_r \in \{-1, 1\}$$

and

$$D_{r'} = -D_r \qquad \text{for } |r - r'| = 1.$$

The matrix $\mathrm{i}\varepsilon D$ is introduced to ensure the existence of the inverse matrix in (7). The latter can easily be seen when we write

$$\hat{H} + \mathrm{i}\varepsilon D = (\hat{H}D + \mathrm{i}\varepsilon \mathbb{1}) D \tag{10}$$

and consider the property

$$(\hat{H}D)^+ = \hat{H}D \tag{11}$$

due to the definition of $\hat{H}$ as a nearest-neighbour matrix (note that $\hat{H}^+ \neq \hat{H}$).

In order to evaluate the averaged internal energy we use the identity

$$(A + \hat{\eta})^{-1} = A^{-1} - A^{-1}(\mathbb{1} + \hat{\eta}A^{-1})_S^{-1} \hat{\eta} A^{-1} \tag{12}$$

where

$$\hat{\eta}_{r\mu,r'\mu'} = \eta I_r \delta_{rr'} \delta_{\mu\mu'}$$

$$I_r = \begin{cases} 1 & \text{for } r \in S \subset \Lambda \\ 0 & \text{for } r \notin S \end{cases}$$

and $(\ )_S^{-1}$ is the inverse on S.

We can perform the integration over V_r when we set

$$A = (\hat{H}D + \mathrm{i}\varepsilon\mathbb{1})|_{V_r} = 0$$

$$\eta = V_r \qquad S = \{r\}$$

since only the 2×2 matrix

$$(\mathbb{1} + \hat{\eta}A^{-1})_S^{-1}\hat{\eta} \tag{13}$$

on the right-hand side of (12) depends on V_r. In particular, this matrix is analytic on the upper (lower) half-plane as a function of V_r if $\varepsilon > 0$ ($\varepsilon < 0$). Thus we obtain with our choice (8) the result

$$\int (\hat{H} + \mathrm{i}\varepsilon D)^{-1} P(V_r)\,\mathrm{d}V_r = \sum_{j=1}^{n} p_j [\hat{H}|_{V_r = a_j + \mathrm{i}\varepsilon(\gamma/|\varepsilon|)D_r} + \mathrm{i}\varepsilon D]^{-1}. \tag{14}$$

We repeat this calculation procedure successively for the other lattice points $r' \in \Lambda$ to find eventually

$$\int (\hat{H} + \mathrm{i}\varepsilon D)^{-1} \prod_{r\in\Lambda} P(V_r)\,\mathrm{d}V_r = \sum_{\{j_r\}} \prod_{r\in\Lambda} p_{j_r} [\hat{H}_0 + V' + \mathrm{i}\varepsilon(1 + \gamma/|\varepsilon|)D]^{-1} \tag{15}$$

with the new discrete random variables

$$V'_{r\mu,r'\mu'} = a_{j_r}\delta_{rr'}\delta_{\mu\mu'}. \tag{16}$$

The expression for $E(T)$ is independent of the original choice of ε, since the imaginary part

$$\begin{aligned} \mathrm{Im} \sum_{r\in\Lambda} \sum_{\mu=1}^{2} (\langle [\hat{H}_0 + V' + \mathrm{i}\varepsilon(1+\gamma/|\varepsilon|)D]^{-1}\rangle_{V'})_{r\mu,r\mu} \\ = -\varepsilon(1+\gamma/|\varepsilon|) \sum_r \sum_\mu (\langle [(\hat{H}_0 + V')(\hat{H}_0^+ + V') + (|\varepsilon| + \gamma)^2]^{-1}\rangle)_{r\mu,r\mu} D_r \end{aligned} \tag{17}$$

vanishes due to the periodic boundary conditions and

$$D_r = -D_{r'}$$

for $|r - r'| = 1$. Then we have the final result

$$E(T) = \lim_{\Lambda\uparrow Z^2} \frac{1}{|\Lambda|} \sum_{r\in\Lambda} \sum_{\mu=1}^{2} (\langle [(\hat{H}_0 + V')(\hat{H}_0^+ + V') + \gamma^2\mathbb{1}]^{-1} (\hat{H}_0^+ + V')\rangle_{V'})_{r\mu,r\mu}. \tag{18}$$

It is remarkable that $\partial^k E(T)/(\partial T)^k$ ($k \geq 0$) is a smooth function of T for any $\gamma > 0$, i.e. the quenched thermodynamic functions derived from the lattice field model do not indicate a phase transition.

This result disagrees with calculations for the model with Gaussian distributed potentials [4, 6]. These authors found, on the basis of perturbative methods, a singular behaviour for the quenched thermodynamic functions at a critical temperature $T_c' < T_c$. However, one can show using non-perturbative methods that the smooth behaviour occurs again at least for a sufficiently large variance $\langle V_r^2\rangle$ of the random potentials (i.e. strong disorder) [7]. The restriction of smoothness to strong disorder in this particular case is perhaps only a consequence of the estimation method which is based on a connected cluster expansion. This speculation is supported by the result presented here, namely we could also construct the connected cluster expansion of $E(T)$ for the model with potentials distributed according to (8). To work with the expansion it must converge absolutely (as we argue for the Gaussian distribution). This condition is satisfied only for γ larger than a certain value γ_0, i.e. for strong disorder. On the other hand, the expression (18) shows the smoothness of $E(T)$ already for $0 < \gamma < \gamma_0$.

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